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Expansions for some confluent hypergeometric functions

Karim M Abadir

Department of Economics, The American University in Cairo, PO Box 2511, Cairo, Egypt

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Abstract. Power series for the parabolic cylinder function, $D_\nu(z)$, are derived from known addition theorems, thus enabling a more compact and efficient expression of the function. One of the expansions is applied to find an asymptotic approximation for the function as $|\nu|\to\infty$ with $|\arg(-\nu)|<\pi$. Next, a large-argument (z) asymptotic addition theorem is derived from an integral representation of $D_\nu(z)$. Before extending this type of summation theorem to the two general confluent hypergeometric functions, motivating mathematical applications to three integration problems are given as examples of the practical usefulness of the formulae.

1. Introduction

One definition of the parabolic cylinder function is

$$D_\nu(z)=2^{\nu/2}\sqrt{\pi}\exp\left(-\frac{z^2}{4}\right)$$
$$\times\left\{{}_1F_1\left(-\frac{\nu}{2};\frac{1}{2};\frac{z^2}{2}\right)\Gamma\left(\frac{1-\nu}{2}\right)^{-1}-\frac{z\sqrt{2}}{\Gamma(-\nu/2)}\,{}_1F_1\left(\frac{1-\nu}{2};\frac{3}{2};\frac{z^2}{2}\right)\right\}\tag{1}$$

(Erdélyi 1953, vol 2, p 117) where Kummer's series is

$${}_1F_1(a;c;x)=\sum_{j=0}^{\infty}\frac{(a)_jx^j}{(c)_jj!}=1+\frac{a}{c}x+\frac{a(a+1)}{c(c+1)}\frac{x^2}{2}+\dots\tag{2}$$

(Erdélyi 1953, vol 1, p 248), and Pochhammer's symbol is $(\alpha)_j=\Gamma(\alpha+j)/\Gamma(\alpha)=\alpha(\alpha+1)\dots(\alpha+j-1)$. For bounded argument z, the function can be calculated as the sum of two power series from (1) and (2). Alternatively, Lebedev (1972, pp 288–9) combines the two series to get

$$D_\nu(z)=\frac{2^{-1-\nu/2}}{\Gamma(-\nu)}\exp\left(-\frac{z^2}{4}\right)\sum_{j=0}^{\infty}\Gamma\left(\frac{j-\nu}{2}\right)\frac{(-z\sqrt{2})^j}{j!}\tag{3}$$

where $\Gamma(-1)/\Gamma(-2)$ should be interpreted as -4, when $\nu+1\in\mathbb{N}$. In the next section, an alternative form for (3) as well as a new power series are presented. Section 3 then considers an asymptotic addition theorem for the parabolic cylinder function. Next, the results of sections 2 and 3 are applied to finding three hitherto unresolved integrals. Finally, asymptotic addition theorems are similarly found for the more general cases of the two confluent hypergeometric functions (equation (2) is one of these functions).

2. The expansions

Addition theorems for (1) have been derived in the case when the argument of the function is bounded (e.g. Erdélyi 1953, vol 2, pp 119 and 124, Erdélyi 1955, vol 3, p 263):

$$\exp\left(-\frac{(x+y)^2}{4}\right)D_\nu(x+y)=\exp\left(-\frac{x^2}{4}\right)\sum_{j=0}^{\infty}\frac{(-y)^j}{j!}D_{\nu+j}(x) \tag{4}$$

$$\exp\left(\frac{(x+y)^2}{4}\right)D_\nu(x+y)=\exp\left(\frac{x^2}{4}\right)\sum_{j=0}^{\infty}\binom{\nu}{j}y^jD_{\nu-j}(x). \tag{5}$$

However, these theorems have not been previously exploited to derive power series for the parabolic cylinder function. To do so, let $x=0$ in (4) and (5), then use $D_\nu(0)=2^{\nu/2}\sqrt{\pi}/\Gamma[(1-\nu)/2]$, the latter part being obtained from the conventional definition of $D_\nu(z)$ in (1) and (2). The resulting new expansions are

$$D_\nu(z)=2^{\nu/2}\sqrt{\pi}\exp\left(\frac{z^2}{4}\right)\sum_{j=0}^{\infty}\frac{(-z\sqrt{2})^j}{j!\Gamma[(1-\nu-j)/2]} \tag{6}$$

$$D_\nu(z)=2^{\nu/2}\sqrt{\pi}\exp\left(-\frac{z^2}{4}\right)\sum_{j=0}^{\infty}\binom{\nu}{j}\frac{(z/\sqrt{2})^j}{\Gamma[(1-\nu+j)/2]}. \tag{7}$$

Expansion (3) could be obtained by applying Legendre's duplication formual to (7) then simplifying by transforming some of the gamma functions therein into ones with arguments of the opposite sign. This explains why (3) obscures the picture when $\nu+1\in\mathbb{N}$ and the expansion—which is then related to the Hermite polynomials—terminates after $1+\text{int}(\nu/2)$ terms. (The $\text{int}(x)$ function returns the largest integer that is less than or equal to x.)

Formulae (6) and (7) are the most compact and numerically efficient series expansions for $D_\nu(z)$ when z is finite. The former becomes more efficient as $|\nu|$ increases. In addition to these advantages of (6) and (7) over the old formulae (1) and (3), one can now derive new approximations for the parabolic cylinder function under certain conditions. For example, using (6), one can find that for z bounded and $|\nu|\to\infty$ with $|\arg(-\nu)|<\pi$

$$\begin{aligned}D_\nu(z)&=2^{\nu/2}\frac{\sqrt{\pi}\exp(z^2/4)}{\Gamma[(1-\nu)/2]}\sum_{j=0}^{\infty}\frac{(-z\sqrt{2})^j\Gamma[(1-\nu)/2]}{j!\Gamma[(1-\nu-j)/2]}\\&\sim 2^{\nu/2}\frac{\sqrt{\pi}\exp(z^2/4)}{\Gamma[(1-\nu)/2]}\sum_{j=0}^{\infty}\frac{(-z\sqrt{1-\nu})^j}{j!}\\&=2^{\nu/2}\frac{\sqrt{\pi}\exp(z^2/4-z\sqrt{1-\nu})}{\Gamma[(1-\nu)/2]}\end{aligned} \tag{8}$$

(cf Abramowitz and Stegun 1965, p 689, Erdélyi 1953, vol 2, p 123). The asymptotic expansion of the ratio of the gamma functions on the first line of (8) is achieved with the help of a formula in Erdélyi (1953, vol 1, p 47).

3. The addition theorem

When the argument $x+y$ is large, (4) and (5) become numerically less efficient and analytically less valuable. The following theorem derives an unconventional addition formula for the parabolic cylinder function with a large argument. It is unconventional because of the asymptotic nature of the sum, which is non-convergent. Nevertheless, *it will be demonstrated that, in spite of its non-convergence, such a formula is not* only numerically efficient for large arguments, but also analytically very useful for *any* value of the argument.

Theorem 1. If x is bounded and $|y|\to\infty$ with $|\arg y|<3\pi/4$, then

$$\exp\left(\frac{(x+y)^2}{4}\right)D_\nu(x+y)=\sum_{j=0}^{n-1}\binom{\nu}{j}y^{\nu-j}\,\mathrm{He}_j(x)+\mathrm{O}(y^{\nu-n}) \tag{9}$$

where $\mathrm{He}_j(x)$ are Hermite's polynomials.

Proof. Consider the integral representation (Gradshteyn and Ryzhik 1980, p 1064)

$$\exp\left(\frac{(x+y)^2}{4}\right)D_\nu(x+y)=\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\left(\frac{2t}{\mathrm{i}}\right)^\nu\exp\left[-2\left(t-\frac{\mathrm{i}(x+y)}{2}\right)^2\right]\mathrm{d}t \tag{10}$$

where $\mathrm{Re}\,\nu>-1$ and $\arg t^\nu=\nu\pi\mathrm{i}$ when $t<0$. Letting $s=t-\mathrm{i}y/2$,

$$\exp\left(\frac{(x+y)^2}{4}\right)D_\nu(x+y)=\sqrt{\frac{2}{\pi}}\exp\left(\frac{x^2}{2}\right)\int_{-\infty-\mathrm{i}y/2}^{\infty-\mathrm{i}y/2}\left(y+\frac{2s}{\mathrm{i}}\right)^\nu\exp(-2s^2+2\mathrm{i}sx)\,\mathrm{d}s$$

$$=\int_{-\infty-\mathrm{i}y/2}^{-\infty}+\int_{-\infty}^{\infty}+\int_{\infty}^{\infty-\mathrm{i}y/2}$$

where the range of integration was deformed by the Cauchy–Goursat theorem since the integrand is analytic in the rectangle with coordinates $(\pm\infty,\mathrm{Im}(-\mathrm{i}y/2))$ and $(\pm\infty,0)$, bounded by the paths of integration. The first and third integrals are zero since *their real paths are unaffected by y and the integrand tends to zero as* $|s|\to\infty$ with $|\arg s|<\pi/4$ or $|\pi-\arg s|<\pi/4$. So

$$\exp\left(\frac{(x+y)^2}{4}\right)D_\nu(x+y)=\sqrt{\frac{2}{\pi}}\exp\left(\frac{x^2}{2}\right)\int_{-\infty}^{\infty}\left(y+\frac{2s}{\mathrm{i}}\right)^\nu\exp(-2s^2+2\mathrm{i}sx)\,\mathrm{d}s.$$

Because of the binomial term, the integrand is multiple valued. Taking the principal value and expanding asymptotically for y, then

$$\exp\left(\frac{(x+y)^2}{4}\right)D_\nu(x+y)$$

$$=\sqrt{\frac{2}{\pi}}\exp\left(\frac{x^2}{2}\right)\sum_{j=0}^{n-1}\binom{\nu}{j}y^{\nu-j}\int_{-\infty}^{\infty}\left(\frac{2s}{\mathrm{i}}\right)^j\exp(-2s^2+2\mathrm{i}sx)\,\mathrm{d}s+R_n$$

where it can be shown along a line of argument similar to that of Whittaker and Watson (1927, pp 342–3) that R_n is bounded and is of $\mathrm{O}(y^{\nu-n})$. By using (10) again, the expansion can be written as

$$\exp\left(\frac{(x+y)^2}{4}\right)D_\nu(x+y)=\exp\left(\frac{x^2}{4}\right)\sum_{j=0}^{n-1}\binom{\nu}{j}y^{\nu-j}D_j(x)+\mathrm{O}(y^{\nu-n})$$

$$=\sum_{j=0}^{n-1}\binom{\nu}{j}y^{\nu-j}\,\mathrm{He}_j(x)+\mathrm{O}(y^{\nu-n}).$$

When $\mathrm{Re}\,\nu=-1$, taking the Cauchy principal value allows the use of the above integral representation as an intermediate tool for the proof. When $\mathrm{Re}\,\nu<-1$, the result follows recursively by differentiating both sides of (9) with respect to y according to Erdélyi (1953, vol 2, equation 8.2(15)) with an initial value of $-1 \leqslant \operatorname{Re} \nu < 0$. This completes the proof of the theorem. □

This result is a generalization of a finite summation theorem given in Erdélyi (1955, vol 3, p 263). His restriction of $\nu + 1 \in \mathbb{N}$ has been removed here. When $\nu + 1 \in \mathbb{N}$, the sum in j is finite and the expansion is exact and valid for any $x + y \in \mathbb{C}$. Otherwise, the expansion is non-convergent and, if used for numerical purposes, only the first few terms should be taken and the point of truncation for the series is decided according to the precision required. The order of the asymptotic approximation is then given by the leading term of the remainder.

Note that (9) encompasses an asymptotic expansion of the parabolic cylinder function given in Erdélyi (1953, vol 2, p 122). Letting $x = 0$ in (9) gives us the asymptotic expansion of the function for $|\arg y| < 3\pi/4$.

4. Applications

In addition to the numerical efficiency considered earlier, the new sums of sections 2 and 3 are useful in solving analytical problems. First, let us consider an application of the new expansion (6). Define the Mellin transform as

$$\mathcal{M}\left\{\exp\left(-\frac{(x+ay)^2}{4}\right)D_\nu(x+ay)\right\} \equiv \int_0^\infty y^\mu \exp\left(-\frac{(x+ay)^2}{4}\right)D_\nu(x+ay)\,\mathrm{d}y \qquad (11)$$

with $\mu > -1$. This integral cannot be found in any of the major published tables (e.g. Abramowitz and Stegun 1965, Gradshteyn and Ryzhik 1980, Oberhettinger 1974, Oberhettinger and Badii 1973). Using (6), it becomes

$$\begin{aligned}\mathcal{M}&\left\{\exp\left(-\frac{(x+ay)^2}{4}\right)D_\nu(x+ay)\right\}\\ &= \sum_{j=0}^{\infty}\frac{(-x)^j}{j!}\int_0^\infty y^\mu \exp\left(-\frac{ay^2}{4}\right)D_{\nu+j}(ay)\,\mathrm{d}y\\ &= \sqrt{\pi}\,(\sqrt{2})^{\nu-\mu-1}a^{-\mu-1}\Gamma(\mu+1)\sum_{j=0}^{\infty}\frac{(-x\sqrt{2})^j}{j!\Gamma(1+(\mu-\nu-j)/2)}\\ &= a^{-\mu-1}\Gamma(\mu+1)\exp(-x^2/4)D_{\nu-\mu-1}(x) \qquad (12)\end{aligned}$$

where the last step follows from (6), and the step before from Oberhettinger (1974, p 145).

The next application involves finding two Laplace transform pairs with the help of theorem 1. Special cases of these results abound in most tables, but the general integrals are nowhere available. Define $\mathscr{L}\{\}$ and $\mathscr{L}^{-1}\{\}$ as the Laplace transform and inverse Laplace transform operators with parameters p and t, respectively. Also, follow Oberhettinger and Badii (1973) in denoting real positive parameters by Latin letters and complex parameters by Greek letters. Then:

(i) $I_1 \equiv \mathscr{L}^{-1}\{\exp(-a\sqrt{p})(b+\sqrt{p})^\nu/\sqrt{p}\}.$

Expanding the binomial term about the branch point $p=0$, and integrating termwise gives an asymptotic expansion similar to (9):

$$I_1=\sum_{j=0}^{\infty}\binom{\nu}{j}b^{\nu-j}\mathscr{L}^{-1}\{\exp(-a\sqrt{p})(\sqrt{p})^{j-1}\}$$

$$=\frac{\exp(-a^2/(8t))}{\sqrt{\pi t}}\sum_{j=0}^{\infty}\binom{\nu}{j}b^{\nu-j}(\sqrt{2t})^{-j}D_j\left(\frac{a}{\sqrt{2t}}\right)$$

(from Oberhettinger and Badii 1973, p 259)

$$=\sqrt{\frac{2}{\pi}}(\sqrt{2t})^{-\nu-1}\exp\left(\frac{ab}{2}+\frac{tb^2}{2}-\frac{a^2}{8t}\right)D_\nu\left(b\sqrt{2t}+\frac{a}{\sqrt{2t}}\right) \qquad \text{(by (9))}.$$

(ii) $$I_2\equiv\mathscr{L}^{-1}\left\{\exp\left[-a\sqrt{p}+\frac{(b+c\sqrt{p})^2}{4}\right]\frac{D_\nu(b+c\sqrt{p})}{\sqrt{p}}\right\}.$$

Expanding the parabolic cylinder function according to (9),

$$I_2=\sum_{j=0}^{\infty}\binom{\nu}{j}c^{\nu-j}\,\mathrm{He}_j(b)\mathscr{L}^{-1}\{(\sqrt{p})^{\nu-j-1}\exp(-a\sqrt{p})\}$$

$$=(\pi t)^{-1/2}\left(\frac{c}{\sqrt{2t}}\right)^{\nu}\exp\left(\frac{b^2}{4}-\frac{a^2}{8t}\right)\sum_{j=0}^{\infty}\binom{\nu}{j}\left(\frac{\sqrt{2t}}{c}\right)^{j}D_j(b)D_{\nu-j}\left(\frac{a}{\sqrt{2t}}\right)$$

$$=(\pi t)^{-1/2}\left(1+\frac{c^2}{2t}\right)^{\nu/2}\exp\left(\frac{b^2}{4}-\frac{a^2}{8t}-\frac{(a-bc)^2}{4(c^2+2t)}\right)D_\nu\left(\frac{ac+2bt}{[2t(c^2+2t)]^{1/2}}\right)$$

where the Laplace inversion is obtained from Oberhettinger and Badii (1973, p 259), and the last expression from the summation theorem in Gradshteyn and Ryzhik (1980, p 1066) subject to the additional condition that Re $\nu\geqslant 0$.

Integrals of this type have proved their worth in solving long-standing problems in mathematical statistics (e.g. see Abadir (1993a, b) for a discussion of and solution to some problems first posed in the 1950s). In particular, I_1 comes up quite frequently when dealing with Brownian motion and Ornstein–Uhlenbeck processes (*op. cit.* for references).

5. Other asymptotic addition theorems

The finite-argument definition of Kummer's function is given in (2), and Tricomi's function is related to it by

$$\Psi(a;c;z)=\frac{\Gamma(1-c)}{\Gamma(1+a-c)}\,{}_1F_1(a;c;z)+\frac{\Gamma(c-1)}{\Gamma(a)}z^{1-c}\,{}_1F_1(a+1-c;2-c;z) \qquad (13)$$

(Erdélyi 1953, vol 1, p 257). Both (2) and (13) are called confluent hypergeometric functions. In this section, asymptotic summation theorems for (2) and (13) will be derived from integral representations of these functions. This method has been used in section 3 above for the parabolic cylinder function, which is a special case of the confluent hypergeometric functions.

Theorem 2. For x bounded and $\operatorname{Re} y \to -\infty$,

$$ {}_1F_1(a;c;x+y)=\Gamma(c)\sum_{j=0}^{n-1}\binom{-a}{j}\frac{(-y)^{-a-j}}{\Gamma(c-a-j)}\,{}_1F_1(-j;c-a-j;x)+O[(-y)^{-a-n}]. $$

Proof. The integral representation (e.g. see Erdélyi 1953, vol 1, p 273, Oberhettinger and Badii 1973, p 239)

$$ {}_1F_1(a;c;t)=\Gamma(c)\mathscr{L}^{-1}\{s^{-c}(1-t/s)^{-a}\} $$

where the Laplace inverse operator ($\mathscr{L}^{-1}\{\}$) has unit parameter, leads to

$$ \begin{aligned} {}_1F_1(a;c;x+y) &= \Gamma(c)\mathscr{L}^{-1}\{s^{-c}(1-(x+y)/s)^{-a}\} \\ &= \Gamma(c)\mathscr{L}^{-1}\left\{\sum_{j=0}^{n-1}\binom{-a}{j}(-y)^{-a-j}s^{a+j-c}(1-x/s)^{j}\right\}+R_n \\ &= \Gamma(c)\sum_{j=0}^{n-1}\binom{-a}{j}(-y)^{-a-j}\mathscr{L}^{-1}\{s^{a+j-c}(1-x/s)^{j}\}+R_n \\ &= \Gamma(c)\sum_{j=0}^{n-1}\binom{-a}{j}\frac{(-y)^{-a-j}}{\Gamma(c-a-j)}\,{}_1F_1(-j;c-a-j;x)+R_n \end{aligned} $$

where R_n can be shown to be bounded and of $O[(-y)^{-a-n}]$ by the method detailed in Whittaker and Watson (1927, pp 342–3). On the first and last expressions of the proof, when either $1-c$ or $1+a-c\in\mathbb{N}$, the expressions are valid in the limit (Erdélyi 1953, vol 1, p 260), thus completing the proof. □

It is clear that the theorem reduces to the asymptotic expansion of Kummer's function for $x=0$ and $\operatorname{Re} y\to-\infty$ (cf Erdélyi 1953, vol 1, p 278). Note that Kummer's function on the right-hand side is a finite sum, thus enhancing the efficiency of the expansion. Finally, the formula is still of use when the argument of Kummer's function is large and positive. In this case, Kummer's transform (Erdélyi 1953, vol 1, p 253) has to be applied to the original function first, in order to transform it into another Kummer's function with an argument of the opposite (negative) sign, thereby making it fit the description of theorem 2. In other words, the repeated application of theorem 2 and Kummer's transform leads to

$$ \begin{aligned} {}_1F_1(a;c;x+y) &\\ &= \Gamma(c)\sum_{j=0}^{n-1}\binom{-a}{j}\frac{(-y)^{-a-j}}{\Gamma(c-a-j)}\,{}_1F_1(-j;c-a-j;x)+O[(-y)^{-a-n}] \\ &\quad+\exp(x+y)\Gamma(c)\sum_{j=0}^{m-1}\binom{a-c}{j}\frac{y^{a-c-j}}{\Gamma(a-j)}\,{}_1F_1(-j;a-j;-x) \\ &\quad+O[y^{a-c-m}\exp(x+y)] \end{aligned} \tag{14} $$

which is the full ($|y|\to\infty$) asymptotic expansion of Kummer's function, thus generalizing the corresponding form given in Erdélyi (1953, vol 1, p 278).

Theorem 3 gives the equivalent result for Tricomi's function, as follows.

Theorem 3. For $b \notin \mathbb{Z}$, y bounded and $\operatorname{Re} x \to \infty$,

$$\Psi(a;\, a-b;\, x+y) = \sum_{j=0}^{n-1} \binom{-a}{j} x^{-a-j} \Psi(-j;\, -j-b;\, y) + O(x^{-a-n})$$
$$= \sum_{j=0}^{n-1} \binom{-a}{j} (b+1)_j x^{-a-j} \, {}_1F_1(-j;\, -j-b;\, y) + O(x^{-a-n}).$$

Proof. Using the relation between the inverse Mellin and inverse Laplace transforms, the integral representation (e.g. Oberhettinger 1974, p 170)

$$\Psi(a;\, a-b;\, t) = \Gamma(-b)\mathscr{L}^{-1}\{s^b(t-s)^{-a}\}$$

where the Laplace inverse operator has unit parameter, is obtained. It leads to

$$\begin{aligned}\Psi(a;\, a-b;\, x+y) &= \Gamma(-b)\mathscr{L}^{-1}\{s^b(x+y-s)^{-a}\} \\ &= \Gamma(-b)\mathscr{L}^{-1}\left\{\sum_{j=0}^{n-1}\binom{-a}{j} x^{-a-j} s^b (y-s)^j\right\} + R_n \\ &= \Gamma(-b)\sum_{j=0}^{n-1}\binom{-a}{j} x^{-a-j}\mathscr{L}^{-1}\{s^b(y-s)^j\} + R_n \\ &= \sum_{j=0}^{n-1}\binom{-a}{j} x^{-a-j}\Psi(-j;\, -j-b;\, y) + R_n \qquad (15)\end{aligned}$$

where R_n can be shown to be bounded and of $O[(-y)^{-a-n}]$ by the method detailed in Whittaker and Watson (1927, pp 342-3). For the second expansion quoted in the theorem, use (13) to substitute for Tricomi's function on the last line of (15). Since $\lim_{\nu\to j} |\Gamma(-\nu)| = \infty$, the result is

$$\begin{aligned}\Psi(a;\, a-b;\, x+y) \\ &= \sum_{j=0}^{n-1}\binom{-a}{j} x^{-a-j}\Big((b+1)_j \, {}_1F_1(-j;\, -j-b;\, y) \\ &\quad + \frac{\Gamma(-j-b-1)}{\Gamma(-j)} y^{j+b+1} \, {}_1F_1(b+1;\, j+b+2;\, y)\Big) + O(x^{-a-n}) \\ &= \sum_{j=0}^{n-1}\binom{-a}{j}(b+1)_j x^{-a-j} \, {}_1F_1(-j;\, -j-b;\, y) + O(x^{-a-n})\end{aligned}$$

which completes the proof. □

Similar remarks apply as before: the theorem reduces to the asymptotic expansion of Tricomi's function when x dominates y (cf Erdélyi 1953, vol 1, p 278), and Kummer's function on the right-hand side is a finite sum which enhances the efficiency of the expansion. Moreover, comparing theorem 2 with the second result of theorem 3 (or the argument of both Laplace inverse transforms in the proofs) gives an insight into the different sign characteristics of the two confluent hypergeometric functions (compare $-y$ with x, and $1/\Gamma(c-a-j)$ with $\Gamma(b+1+j)$ in theorems 2 and 3, respectively).

The theorems derived in this section can help evaluate confluent hypergeometric functions when the argument is large. In addition, they can help in the derivation of closed forms for seemingly difficult integrals. For example, entry 2.4.17 of Oberhettinger and Badii (1972, p 239) may be easily derived by means of the simpler entry 2.4.18 and theorem 2.

6. Conclusion

The application of the new formulae presented here is by no means restricted to finding out new integrals. They can also be used for other purposes such as efficient numerical computations, deriving new asymptotic approximations as in (8), or evaluating series of some transcendental functions which possess integral representations involving parabolic cylinder functions. The techniques presented here can also be extended to other transcendental functions to find new expansions for them.

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